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Rational Approximation to e^x and to Related Functions

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According to a well-known result of S. N. Bernstein, e^x can be approximated uniformly on $[-1, 1]$ by polynomials of degree $\leqslant n$ with an error of the order $[2^n(n+1)!]^{-1}$. In this note it is shown that the smallest (uniform norm) error in approximating e^x by reciprocals of polynomials of degree $\leqslant n$ is also of the order $[2^n(n+1)!]^{-1}$. We denote throughout by $P_n(x)$, $Q_n(x)$ real polynomials of degree $\leqslant n$. We show, furthermore, that the smallest error in approximating e^x by rational functions of the form $P_n(x)/Q_n(x)$ where the coefficients of Q_n are $\geqslant 0$ is again of that same order.

INTRODUCTION

It is known that the smallest uniform error obtained in approximating e^x on $[-1, 1]$ by polynomials of degree $\leqslant n$ is of the order $[2^n(n+1)!]^{-1}$. If one analyzes this result carefully, then the following questions arise naturally.

Q.1. How close can one approximate e^x on $[-1, 1]$ by reciprocals of polynomials of degree $\leqslant n$?

Q.2. How close can one approximate e^x on $[-1, 1]$ by rational functions of the form $P_n(x)/Q_n(x)$ where the coefficients of Q_n are $\geqslant 0$?

Q.3. How close can one approximate e^x on $[-1, 1]$ by general rational functions of the form $P_m(x)/Q_n(x)$?

In this note we answer these questions.

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Set

$$E_n(e^x) = \inf_{P \in \pi_n} \| e^x - P \|_{L_\infty[-1,1]},$$

$$E_{0,n}(e^x) = \inf_{P \in \pi_n} \| e^x - P^{-1} \|_{L_\infty[-1,1]},$$

$$E_n^*(e^x) = \inf_{r \in \pi_n^*} \| e^x - r \|_{L_\infty[-1,1]},$$

where π_n denotes the class of all real polynomials of degree at most n and π_n^* denotes the class of all rational functions of the form $P_n(x)/Q_n(x)$, $Q_n(x)$ having nonnegative coefficients only.

Recently one of us [6] proved

THEOREM A. *There is a rational function of the form* $P_m(x)/Q_n(x)$ *for which*

$$\left| e^x - \frac{P_m(x)}{Q_n(x)} \right|_{L_\infty[-1,1]} \leqslant \frac{(\text{const})2^{-m-n}(m!)(n!)}{(m+n)!\,(m+n+1)!}. \tag{1}$$

In this note we prove the following results:

THEOREM B. *For every* $P_m(x)$, $Q_n(x)$,

$$\left| e^x - \frac{P_m(x)}{Q_n(x)} \right|_{L_\infty[-1,1]} \geqslant \frac{2^{-2n-2m-4}e^{-(2n+m+3)/2(m+n+2)}}{(3+2(2^{1/2}))^n(m+n+2)^{m+n+2}}. \tag{2}$$

THEOREM C. *For every* $P_n(x)$ $(n \geqslant 2)$,

$$\| e^x - [P_n(x)]^{-1} \|_{L_\infty[-1,1]} \geqslant \frac{(4-e)}{4e^3 2^n (n+1)!}. \tag{3}$$

THEOREM D. *Given* P_n, Q_n, *the latter having nonnegative coefficients only, we have*

$$\left| e^x - \frac{P_n(x)}{Q_n(x)} \right|_{L_\infty[-1,1]} \geqslant \frac{1}{2^n(n+1)!\,(2^{1/2})}. \tag{4}$$

Remarks. The method used to prove Theorem B is a refinement of that used in [5]. Unfortunately, this method does not work to prove Theorems C and D.

We need four lemmas.

LEMMA 1 [4, p. 78]. *Let a real function* $f(x)$ *be* $(n+1)$ *times continuously differentiable in* $[-1, 1]$. *Then there exists a number* ξ, $-1 < \xi < 1$, *such that*

$$E_n(f) = \frac{|f^{(n+1)}(\xi)|}{2^n(n+1)!}.$$

Notice that the above stated result of S. N. Bernstein follows from Lemma 1.

LEMMA 2 [7, p. 68]. *If* $\max_{a\leqslant x\leqslant b} |P_n(x)| \leqslant 1$, *then for* $x < a$ *and* $x > b$,

$$|P_n(x)| \leqslant \left| T_n\left(\frac{2x-a-b}{b-a}\right)\right|,$$

where

$$2T_n(x) = (x + (x^2-1)^{1/2})^n + (x - (x^2-1)^{1/2})^n.$$

LEMMA 3. *Let* Δ *denote the difference operator with increment* 1. *Then*

$$\Delta^{m+1}((A+1)^x Q(x)) = (A+1)^x ((A+1)\Delta + A)^{m+1} Q(x).$$

Proof. It is well known [3, p. 97, (10)] that

$$\Delta^n((A+1)^x Q(x)) = \sum_{i=0}^{n} \binom{n}{i} \Delta^i Q(x)\, \Delta^{n-i} E^i (A+1)^x, \tag{*}$$

where $E = 1 + \Delta$.

A little computation based on (*) and the fact that

$$\Delta^m f(x) = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} f(x+k)$$

gives the result.

LEMMA 4 [3, p. 13]. *If* $f(x)$ *is a polynomial of degree at most* n, *then*

$$(1+\Delta)^{-n} f(x) = \sum_{i=0}^{n} (-1)^i \binom{n+i-1}{i} \Delta^i f(x).$$

Proof of Theorem B. We prove here for convenience that for each $A > 0$, and with $\alpha = m + n + 2$,

$$\left\| (A+1)^x - \frac{P_m(x)}{Q_n(x)} \right\|_{L_\infty[-\alpha,\alpha]} \geqslant \frac{A^{m+n+2} 2^{-2n-2m-4}}{(3 + 2(2)^{1/2})^n (A+1)^{n+1+\alpha}}. \tag{5}$$

Set

$$\epsilon = \left\| (A+1)^x - \frac{P_m(x)}{Q_n(x)} \right\|_{L_\infty[-\alpha,\alpha]}. \tag{6}$$

Normalize $Q_n(x)$ so that

$$\operatorname*{Max}_{[-\alpha,0]} |Q(x)| = 1. \tag{7}$$

Then by Lemma 2,

$$\max_{[-\alpha,\alpha]} | Q(x)| \leqslant (3 + 2(2)^{1/2})^n. \tag{8}$$

From (6) and (8),

$$\|(A + 1)^x Q_n(x) - P_m(x)\|_{L_\infty[-\alpha,\alpha]} \leqslant \epsilon(3 + 2(2^{1/2}))^n. \tag{9}$$

Set $R(x) = Q_n(x)(A + 1)^x - P_m(x)$. Then by Lemma 3,

$$\begin{aligned}
\Delta^{m+1}R(x) &= \sum_{i=0}^{m+1} \binom{m+1}{i} (-1)^{m+1-i} R(x + i) \\
&= \Delta^{m+1}[(A + 1)^x Q_n(x) - P_m(x)] \\
&= \Delta^{m+1}[Q_n(x)(A + 1)^x] \\
&= (A + 1)^x ((A + 1) \Delta + A)^{m+1} Q_n(x).
\end{aligned} \tag{10}$$

From (10) we get, for $-\alpha \leqslant x \leqslant n + 1$,

$$|((A + 1) \Delta + A)^{m+1} Q_n(x)| \leqslant (A + 1)^{-x} 2^{m+1} \epsilon(3 + 2(2^{1/2}))^n. \tag{11}$$

Set

$$S(x) = ((A + 1) \Delta + A)^{m+1} Q_n(x).$$

Then, for $-\alpha \leqslant x \leqslant 0$, by Lemma 4 and (11),

$$\begin{aligned}
| Q_n(x)| &\leqslant |((A + 1)\Delta + A)^{-m-1} S(x)| \\
&\leqslant A^{-m-1} \left|\left(1 + \frac{(A + 1)}{A} \Delta\right)^{-m-1} S(x)\right| \\
&\leqslant A^{-m-1} \left(\frac{A + 1}{A}\right)^{n+1} \sum_{i=0}^{n+1} \binom{m+i}{i} \sum_{l=0}^{i} \binom{i}{l} | S(x + l)| \\
&\leqslant \epsilon 2^{m+1}(3 + 2(2^{1/2}))^n A^{-m-n-2}(A + 1)^{n+1+\alpha} 2^{n+1} \sum_{i=0}^{n+1} \binom{m+i}{i} \\
&\leqslant \epsilon 2^{m+n+2}(3 + 2(2^{1/2}))^n A^{-m-n-2} (A + 1)^{n+1+\alpha} 2^{m+n+2}.
\end{aligned} \tag{12}$$

Hence, at a point $x \in [-\alpha, 0]$, we get from (12) and (7),

$$\epsilon \geqslant A^{m+n+2}\, 2^{-2n-2m-4}(3 + 2(2)^{1/2})^{-n} (A + 1)^{-n-1-\alpha}. \tag{13}$$

(5) follows from (13), and Theorem B follows from (5), by choosing

$$(A + 1) = \exp\left(\frac{1}{(m + n + 2)}\right).$$

COROLLARY. *For any $P_m(x)$, $Q_n(x)$,*

$$\left\| e^x - \frac{P_m(x)}{Q_n(x)} \right\|_{L_\infty[a,b]} \geqslant \frac{(b-a)^{m+n+2} e^{(a+b)/2(m+n+2)}}{8^{m+n+2}(3+2(2^{1/2}))^n e^{(b-a)(n+1+\alpha)/2(m+n+2)}}.$$

Proof. From (5),

$$\left\| (A+1)^y - \frac{P(y)}{Q(y)} \right\|_{L_\infty[-\alpha,\alpha]} \geqslant \frac{A^{m+n+2} 2^{-2n-2m-4}}{(3+2(2^{1/2}))^n (A+1)^{n+1+\alpha}}. \qquad (5')$$

Set

$$y = \frac{\alpha(2x-a-b)}{b-a}, \qquad (*)$$

so that $y \in [-\alpha, \alpha] \to x \in [a, b]$.
From (5′) and (*),

$$\left\| (A+1)^{2x\alpha(b-a)^{-1}} - \frac{P(x)}{Q(x)} \right\|_{L_\infty[a,b]} \geqslant \frac{(A+1)^{((a+b)/(a-b))} A^{m+n+2} 2^{-2n-2m-4}}{(3+2(2^{1/2}))^n (A+1)^{n+1+\alpha}}. \quad (**)$$

Now choose

$$(A+1) = \exp\left(\frac{b-a}{2\alpha}\right); \qquad (***)$$

then the result follows from (**).

Proof of Theorem C. For a $P_n(x)$ and $[-1, 1]$, set

$$\epsilon = \left\| e^x - \frac{1}{P_n(x)} \right\|. \qquad (14)$$

Then on $[-1, 1]$,

$$| P_n(x) - e^{-x} | \leqslant \epsilon e^{-x} | P_n(x)| \leqslant \frac{\epsilon \, | P_n(x)|}{e^{-1}}. \qquad (15)$$

Also, on $[-1, 1]$,

$$\left| \frac{1}{P_n(x)} \right| \geqslant e^x - \epsilon \geqslant \frac{1}{e} - \epsilon. \qquad (16)$$

From Theorem A, we have for the case $m = 0$, and all $n \geqslant 2$,

$$\epsilon \leqslant \frac{1}{4}. \qquad (17)$$

From (16) and (17),

$$\left| \frac{1}{P_n(x)} \right| \geqslant \frac{1}{e} - \frac{1}{4} = \frac{4-e}{4e},$$

i.e.,

$$\operatorname*{Max}_{[-1,1]} | P_n(x)| \leqslant \frac{4e}{4-e}. \tag{18}$$

From (15) and (18),

$$| P_n(x) - e^{-x} | \leqslant \frac{4\epsilon}{4-e}. \tag{19}$$

By Lemma 1 and (19),

$$\frac{4\epsilon e^2}{4-e} \geqslant E_n(e^{-x}) \geqslant \frac{1}{e2^n(n+1)!}. \tag{20}$$

From (20) we have the required result.

Proof of Theorem D. Set

$$\delta = \left\| e^x - \frac{P_n(x)}{Q_n(x)} \right\|_{L_\infty[-1,1]}. \tag{21}$$

As before, on $[-1, 1]$,

$$| e^x Q(x) - P_n(x)| \leqslant \delta \sum_{k=0}^{n} a_k, \tag{22}$$

where $Q_n(x) = \sum_{k=0}^{n} a_k x^k > 0$ on $[-1, 1]$, and $a_k \geqslant 0$. Set

$$f(x) = e^x Q_n(x) = \sum_{k=0}^{\infty} b_k x^k = \frac{A_0}{2} + \sum_{k=0}^{\infty} A_k T_k(x),$$

where A_k denote the Fourier–Chebychev coefficients. Since all b_k are $\geqslant 0$, we have [1, p. 116],

$$A_{n+1} = \frac{1}{2^n} \left[b_{n+1} + \frac{n+3}{2^2} b_{n+3} + \frac{(n+4)(n+5)}{2^4 2!} b_{n+5} + \cdots \right]. \tag{23}$$

Further, it is known [1, p. 111] that

$$E_n(f(x)) \geqslant (A_{n+1}^2 + A_{n+2}^2 + A_{n+3}^2 + \cdots)^{1/2} 2^{-1/2}. \tag{24}$$

Hence,

$$E_n(f(x)) \geqslant \frac{A_{n+1}}{2^{1/2}}. \tag{25}$$

From (22), (23), and (25), we have

$$\delta \sum_{k=0}^{n} a_k \geqslant \| e^x Q_n - P_n \|_{L_\infty[-1,1]} \geqslant \frac{A_{n+1}}{2^{1/2}} . \tag{26}$$

A simple calculation gives

$$b_{n+1} = \frac{a_0}{(n+1)!} + \frac{a_1}{n!} + \frac{a_2}{(n-1)!} + \cdots + a_n \geqslant \frac{1}{(n+1)!} \left(\sum_{k=0}^{n} a_k \right). \tag{27}$$

Hence, from (26), (27), and (23),

$$\delta \sum_{k=0}^{n} a_k \geqslant \frac{A_{n+1}}{2^{1/2}} \geqslant \frac{\sum_{k=0}^{n} a_k}{2^n (2)^{1/2} (n+1)!} , \tag{28}$$

i.e.,

$$\delta \geqslant \frac{1}{2^n (n+1)! (2^{1/2})} .$$

Remarks on Theorems A–D. By Bernstein's result, we have

$$\lim_{n \to \infty} \{(n+1)!\, E_n(e^x)\}^{1/n} = 2^{-1}. \tag{29}$$

From Theorem A, with $m = 0$,

$$E_{0,n}(e^x) \leqslant \frac{\text{const.} \cdot 2^{-n}}{(n+1)!} . \tag{30}$$

From Theorem C,

$$E_{0,n}(e^x) \geqslant \frac{(4-e)}{4e^3 2^n (n+1)!} . \tag{31}$$

From (30) and (31),

$$\lim_{n \to \infty} \{(n+1)!\, E_{0,n}(e^x)\}^{1/n} = \tfrac{1}{2} . \tag{32}$$

From Theorem D,

$$E_n^*(e^x) \geqslant \frac{1}{2^{n+1/2} (n+1)!} . \tag{33}$$

On the other hand, by Lemma 1,

$$E_n^*(e^x) \leqslant \frac{e}{2^n (n+1)!} . \tag{34}$$

Hence by (33) and (34) we have

$$\lim_{n\to\infty}\{(n+1)\, E_n^*(e^x)\}^{1/n} = 2^{-1}. \tag{35}$$

Thus, from (29), (32), and (35),

$$\begin{aligned}\lim_{n\to\infty}\{(n+1)!\, E_n(e^x)\}^{1/n} &= \lim_{n\to\infty}\{(n+1)!\, E_{0,n}(e^x)\}^{1/n}\\ &= \lim_{n\to\infty}\{(n+1)!\, E_n^*(e^x)\} = 2^{-1}.\end{aligned} \tag{36}$$

(36) shows that for e^x, E_n, $E_{0,n}$, and E_n^* are of the same order. Does this holds more generally? We shall show that this phenomenon fails for $e^x - e^{-1}$ and for e^{-x}.

From the above results it is clear that for $e^x - e^{-1}$ the formulas for E_n and E_n^* are the same as in (36). It can also be shown easily [2, p. 391] that $e^x - e^{-1}$ can be approximated uniformly by reciprocals of polynomials of degree n as close as we wish. But $E_{0,n}(e^x - e^{-1})$ is not known. We prove here the following

THEOREM E. *Every $P_n(x)$ $(n \geqslant 3)$ satisfies*

$$\left\| e^x - e^{-1} - \frac{1}{P_n(x)} \right\|_{L_\infty[-1,1]} \geqslant \frac{1}{28n^2}\,. \tag{37}$$

Proof. Set

$$\delta = \left\| e^x - e^{-1} - \frac{1}{P_n(x)} \right\|_{L_\infty[-1,1]}\,. \tag{38}$$

Then we have over the interval $[n^{-2} - 1, 1]$, from (38),

$$\left| \frac{1}{P(x)} \right| \geqslant e^x - \frac{1}{e} - \delta \geqslant \frac{e^{n^{-2}}}{e} - \frac{1}{e} - \delta \geqslant e^{-1}n^{-2} - \delta. \tag{39}$$

Case (i). If $1/en^2 - \delta \leqslant 0$, then

$$\delta \geqslant \frac{1}{en^2}\,. \tag{40}$$

Case (ii). If $1/en^2 - \delta > 0$, then from (39),

$$\operatorname*{Max}_{(n^{-2}-1,1)} |P(x)| \leqslant \frac{en^2}{1 - en^2\delta}\,. \tag{41}$$

By applying Lemma 2 to (41), we obtain

$$|P(0)| \leqslant \operatorname*{Max}_{[-1,1]} |P(x)| \leqslant \frac{25n^2}{1 - en^2\delta}. \tag{42}$$

On the other hand, we get from (38)

$$1/\delta \leqslant |P(0)|. \tag{43}$$

From (42) and (43),

$$\delta \geqslant 1/28n^2. \tag{44}$$

Equation (37) follows from (40) and (44).

Clearly $E_n(e^{-x}) = E_n(e^x)$, $E_{0,n}(e^{-x}) = E_{0,n}(e^x)$. But as we shall see, $E_n^*(e^{-x})$ is much smaller.

THEOREM F. *There is a rational function of degree n and of the form*

$$r(x) = \frac{P_n(x)}{Q_n(x)} = \frac{\int_0^\infty t^n(t - 1 - x)^n e^{-t}\, dt}{\int_0^\infty t^n(t + 1 + x)^n e^{-t}\, dt}, \tag{45}$$

for which we have, for all $n \geqslant 2$,

$$\| e^{-x} - er(x)\|_{L_\infty[-1,1]} \leqslant \frac{e}{(2n)!}. \tag{46}$$

Proof. It is easy to check that for $-1 \leqslant x \leqslant 1$,

$$|r(x) - e^{-x-1}| = \left| \frac{\int_0^\infty t^n(t - x - 1)^n e^{-t}\, dt - \int_0^\infty t^n(t + 1 + x)^n e^{-(t+x+1)}\, dt}{\int_0^\infty t^n(t + 1 + x)^n e^{-t}\, dt} \right|$$

$$= \left| \frac{\int_0^\infty t^n(t - x - 1)^n e^{-t}\, dt - \int_{1+x}^\infty (t - x - 1)^n t^n e^{-t}\, dt}{\int_0^\infty t^n(t + 1 + x)^n e^{-t}\, dt} \right| \tag{47}$$

$$= \frac{\left| \int_0^{1+x} t^n(x + 1 - t)^n e^{-t}\, dt \right|}{\left| \int_0^\infty t^n(t + 1 + x)^n e^{-t}\, dt \right|} = \frac{I_1}{I_2}.$$

By observing the known fact that

$$t(x + 1 - t) \leqslant \frac{(1 + x)^2}{4},$$

we get for all $x \in [-1, 1]$

$$I_1 \leqslant \int_0^\infty e^{-t}\, dt. \tag{48}$$

On the other hand, we have for $-1 \leqslant x \leqslant 1$,

$$I_2 \geqslant \int_0^\infty t^{2n} e^{-t}\, dt = (2n)!. \tag{49}$$

Equation (46) follows from (47), (48), and (49).

Open Problems

Q.1. Is it possible to approximate e^x on $[-1, 1]$ by polynomials of degree n having nonnegative coefficients only with an error better than the one obtained by $\sum_{k=0}^{n} x^k/k!$?

Q.2. How close can one approximate e^x on $[-1, 1]$ by monotone polynomials of degree $\leqslant n$?

Q.3. How close can one approximate e^x on $[-1, 1]$ by the ratio of two monoton polynomials of degree $\leqslant n$?

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